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# Some Consequences of the Nonlocalizability of Space-Time Events

**K. Ruebenbauer**[1]

*Department of Chemistry, McMaster University, 1280 Main Street West, Hamilton, Ontario, Canada L8S 4M1*

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Space-time events are characterized by their coordinates $x$ from the classical point of view. The same events from the quantum-mechanical point of view should be described rather by the expectation value of coordinates $\langle X \rangle$. The expectation value could be evaluated by introducing a density operator $\rho(x,x')$ associated with the event. In the case where $\rho(x,x')$ cannot be described by delta functions strictly monochromatic radiation does not exist. If localizability is limited by Planck's length there are no narrower spectral lines than $\simeq 2\times 10^{-29}\,E^2$ (eV) where $E$ stands for the photon energy.

The space-time event $P(x,\alpha)$ is completely described from the classical point of view by spatial and temporal coordinates $x=\{\bar{r},t\}$, where $\bar{r}=\{r_1,r_2,r_3\}$ and by a set of parameters $\alpha=\{\alpha_s\}$ not explicitly dependent on $x$, i.e., $\partial\alpha_s/\partial x_\mu \equiv 0$, where $x=\{x_\mu\}$ and $\mu=1,2,3,4$. The same event considered from the quantum-mechanical point of view should be described by replacing $x$ by the operator $\hat{X}$ and using rather an expectation value $\langle X \rangle$ in place of the classical coordinates $x$. The expectation value of the operator $\hat{X}$ could generally be evaluated in the following way:

$$\langle X \rangle = \mathrm{Tr}(\hat{\rho}\hat{X}) \tag{1}$$

where $\hat{\rho}=\{\rho(x,x';\alpha)\}$ stands for the density operator associated with the event $P$ [and therefore $\mathrm{Tr}(\hat{\rho})=1$] and matrix elements of the operator $\hat{X}$ may be defined as follows:

$$X(x,x') = x\delta^4(x-x') \tag{2}$$

[1]On leave from the Institute of Nuclear Physics, Kraków, ul. Radzikowskiego 152, Poland.



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It is obvious that the each point $x$ of space-time is associated with a separate basis vector in the Hilbert space of the operators $\hat{\rho}$ and $\hat{X}$. Matrix elements of the operator $\hat{A}=\hat{\rho}\hat{X}$ have the form

$$A(x,x')=\int d^4x''\rho(x,x'')X(x'',x')=x'\rho(x,x') \quad (3)$$

and

$$\langle X\rangle=\int d^4x\,x\rho(x,x)=\int d^4x\,x\rho(x) \quad (4)$$

where $\rho(x)\geqslant 0$ and $\int d^4x\rho(x)=1$.

For relatively narrow distribution functions $\rho(x,\alpha)$ it is practical to evaluate $\rho(x,\alpha)$ in terms of the distributional moments. The moments are expressed as follows:

$$g_{l_1l_2l_3l_4}(\langle X\rangle,\alpha)=\int d^4\xi\,\xi_1^{l_1}\xi_2^{l_2}\xi_3^{l_3}\tau^{l_4}\rho(\xi,\alpha) \quad (5)$$

where $\xi=x-\langle X\rangle=\{\bar{\xi},\tau\}=\{\xi_1,\xi_2,\xi_3;\tau\}$ and $\{l_1,l_2,l_3,l_4\}=0,1,\ldots$. It is obvious that $g_{0000}\equiv 1$ and $g_{l_1l_2l_3l_4}\equiv 0$ for any $l_\mu=1$. If $\rho(\xi)=\delta^4(\xi)$ all $g_{l_1l_2l_3l_4}\equiv 0$ except $g_{0000}$ and the classical description is restored, otherwise complete physical information is contained in the moments.

The density operator $\hat{\rho}$ could be evaluated also in the energy-momentum representation—namely,

$$\rho(k,k';\alpha)\sim\int d^4\xi d^4\xi'\rho(\xi,\xi';\alpha)e^{i(k\xi-k'\xi')}, \qquad \rho(k,k)\geqslant 0 \quad (6)$$

where $k=\{\bar{k},-\omega\}=\{k_1,k_2,k_3;-\omega\}$ describes the wave vector. The variables $\bar{k}$ and $\omega$ are not independent but connected by the dispersion relations. For a free particle

$$\bar{k}=\bar{\kappa}(\omega/c)[1+2(\omega_0/\omega)]^{1/2} \quad (7)$$

where $\bar{\kappa}=\bar{k}/(\bar{k}\cdot\bar{k})^{1/2}$ and $\omega_0=(m_0c^2)/\hbar$, $m_0$ being the particle rest mass and $\omega\geqslant 0$. Therefore

$$\rho(k,k')=\rho(\bar{\kappa},\bar{\kappa}';\omega,\omega') \quad (8)$$

For localizable events $\rho(\xi,\xi')=\delta^4(\xi)\delta^4(\xi')$ and $\rho(k,k')=\text{const}$ but otherwise $\lim_{\omega\to\infty}\rho(\omega,\omega)=0$. On the other hand $\rho(\omega,\omega)\simeq\text{const}$ for $g_{0002}\omega^2\ll 1$.

The uncertainty $\Delta T$ of the time interval $T=|t-t'|$ approximately satisfies the relation $\Delta T=(2g_{0002})^{1/2}$ provided that $\partial g_{l_1l_2l_3l_4}/\partial\langle X_\mu\rangle\equiv 0$ and $T\gg(g_{0002})^{1/2}$. The uncertainty $\Delta\omega$ for the inverse time interval $\omega=2\pi/T$ follows approximately the equation $\Delta\omega=|\partial\omega/\partial T|\Delta T$ and hence $\Delta\omega=\omega^2[g_{0002}/(2\pi^2)]^{1/2}$. For almost monochromatic radiation ($m_0=0$) the equation $\omega=E/\hbar$ is rather well satisfied, where $E$ stands for the average energy of the particle within the beam and $\omega$ describes the frequency of the radiation. Hence, the energy spread $\Delta E$ within the beam has the approximate form

$$\Delta E=E^2\left[g_{0002}/(2\pi^2\hbar^2)\right]^{1/2} \tag{9}$$

and the relative dimensionless spread $\delta E=\Delta E/E$ has the form

$$\delta E=E\left[g_{0002}/(2\pi^2\hbar^2)\right]^{1/2} \tag{10}$$

For a real space-time almost devoid of matter it seems reasonable to identify $g_{0002}$ with the square of Planck's time interval $(G\hbar)/c^5$, where $G$ stands for the gravitational constant (Wheeler, 1962; Sakharov, 1968; and Kuchař, 1970). In such a case

$$\delta E=E\left[G/(2\pi^2\hbar c^5)\right]^{1/2}\simeq 2\times 10^{-29}\,E\ (\text{eV}) \tag{11}$$

For a space-time where the density operator associated with the event coordinates has matrix elements different from $\delta^4(x-\langle X\rangle)\delta^4(x'-\langle X\rangle)$ strictly monochromatic radiation does not exist.

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